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Asymptotic degeneracy of dyonic N=4 string states and black hole entropy

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ABSTRACT

It is shown that the asymptotic growth of the microscopic degeneracy of BPS dyons in four-dimensional $N = 4$ string theory captures the known corrections to the macroscopic entropy of four-dimensional extremal black holes. These corrections are subleading in the limit of large charges and originate both from the presence of interactions in the effective action quadratic in the Riemann tensor and from non-holomorphic terms. The presence of the non-holomorphic corrections and their contribution to the thermodynamic free energy is discussed. It is pointed out that the expression for the macroscopic entropy, written as a function of the dilaton field, is stationary at the horizon by virtue of the attractor equations.

1 Introduction

String theory predicts deviations from the Bekenstein-Hawking area law for the entropy of black holes. For large charges the microstate counting yields a statistical entropy which generically coincides with the area of the corresponding macroscopic black hole solutions [1]. In a number of cases, also subleading corrections can be determined. This is especially relevant for the heterotic string, where certain elementary string states can be identified with black holes. For 1/2-BPS states the charges are restricted and as a result the leading contributions to the entropy vanish [2]. Therefore, the dominant contribution to area and entropy will come from subleading terms, which do not obey the proportionality relation as implied by the area law. In all of this it is important that one is dealing with BPS states, corresponding to extremal black holes, so that the effects of string interactions remain under control.

On the macroscopic side subleading corrections have been extensively studied in the framework of four-dimensional $N = 2$ supergravity, where they are induced by couplings in the effective Wilsonian action that are quadratic in the curvature tensor. In [3] a general formula for the entropy was presented in terms of the homogeneous, holomorphic function $F(X, A)$, in which these $N = 2$ supergravity Lagrangians are encoded. Here the holomorphic variables X are related to the complex moduli associated with vector supermultiplets, and the dependence on A characterizes the terms quadratic in the Riemann tensor. The $N = 2$ entropy formula was successfully confronted with the results from microstate counting for a class of black holes arising in compactifications of M-theory and type-IIA string theory [4,5]. In the M-theory setting the microscopic object that enters is the five-brane, wrapped on a four-cycle of a Calabi-Yau three-fold. An important role in the supergravity analysis is played by the attractor equations [6–8] which fix the values of the moduli at the horizon in terms of the black hole charges. For effective actions with interactions quadratic in the curvature the validity of these attractor equations was established in [9].

Recently, it was shown [10] that the $N = 2$ entropy formula can be rewritten as a Legendre transform of a real function, $\mathcal{F}(Q_{\rm m}, \phi)$, where ϕ denotes the electric potentials at the horizon and the electric charges are given by $Q_{\rm e} = \partial\mathcal{F}/\partial\phi$. Subsequently $\mathcal{F}$ was identified with the logarithm of a mixed black hole partition function, which is microcanonical with respect to the magnetic charges $Q_{\rm m}$ and canonical with respect to the electric potentials ϕ at the horizon. It was then conjectured that this partition function can be written as a Laplace transform of the microscopic black hole degeneracies $d(Q_{\rm e}, Q_{\rm m})$. The original result of [3] is to be recovered in the limit of large electric charges. These observations have rekindled the interest in the question of how the entropy formula is precisely related to the actual microscopic degeneracies. In this paper we will study the relation between the entropy formula and the microscopic degeneracies for $N = 4$ dyons proposed in [11] beyond the leading order.

In [12] the modified entropy formula was already applied to heterotic black holes. Although the formula was initially derived for $N = 2$ supergravity, the result can readily be generalized to the case of heterotic $N = 4$ supersymmetric compactifications. This involves

an extension of the target-space duality group from SO(2,18) to SO(6,22) with a corresponding extension of the charges and the moduli. The $N=4$ supersymmetric heterotic models have dual realizations as type-II string compactifications on $K3 \times T^2$. In contrast to $N=2$ Calabi-Yau compactifications, the holomorphic function which encodes the effective Wilsonian action is severely restricted in the $N=4$ case. Therefore it is often possible to obtain exact predictions in this context. In [12] the perturbative holomorphic function for the $N=4$ heterotic theory was appropriately extended in order to obtain results that were invariant under both target-space duality and S-duality. While the first requirement posed no particular problems, the latter necessitates the addition of non-holomorphic terms. This feature is not unexpected: the Wilsonian couplings are holomorphic but may not fully reflect the symmetries of the underlying theory, while the physical couplings must reflect the symmetry and may thus have different analyticity properties. It turned out that the non-holomorphic terms are determined uniquely by requiring S-duality and consistency with string perturbation theory, and are in accord with the results of [13].

Including the non-holomorphic corrections, the result of [12] can be summarized as follows. The non-trivial attractor equations are the ones that determine the horizon value of the complex dilaton field S in terms of the black hole charges. They read as follows,

$$\begin{aligned} |S|^2\,Q_{\rm m}^2 &= Q_{\rm e}^2 + \frac{128\,c_1}{\pi}(S+\bar S)\left(S\frac{\partial}{\partial S}+\bar S\frac{\partial}{\partial \bar S}\right)\log\left[(S+\bar S)^6|\eta(S)|^{24}\right]\ , \\ (S-\bar S)\,Q_{\rm m}^2 &= -\,2\,{\rm i}\,Q_{\rm e}\!\cdot Q_{\rm m} - \frac{128\,c_1}{\pi}(S+\bar S)\left(\frac{\partial}{\partial S}-\frac{\partial}{\partial \bar S}\right)\log\left[(S+\bar S)^6|\eta(S)|^{24}\right]\ . \end{aligned} \tag{1}$$

Here $Q_{\rm e}^2$, $Q_{\rm m}^2$ and $Q_{\rm e}\!\cdot Q_{\rm m}$ are the three target-space duality invariant contractions of the electric and the magnetic charges, $Q_{\rm e}$ and $Q_{\rm m}$, which, in the $N=4$ extension, take values in the lattice $\Gamma^{6,22}$. From the supergravity calculations there is no intrinsic definition of the lattice of charges and consequently the dilaton normalization is a priori not known.[1] Our definition of the Dedekind eta-function $\eta(S)$ follows from the asymptotic formula, $\log\eta(S)\approx -\frac{1}{12}\pi\,S + {\rm e}^{-2\pi S} + \mathcal{O}({\rm e}^{-4\pi S})$. We also recall that $\eta^{24}(S)$ is a modular form of degree 12, so that $\eta^{24}(S') = ({\rm i}\,c\,S+d)^{12}\,\eta^{24}(S)$, where S' is the transformed dilaton field, given below in (3). The constant c_1 must be equal to $c_1=-\frac{1}{64}$, as we shall discuss later.

The expression for the macroscopic entropy reads,

$$\mathcal{S}_{\rm macro} = -\pi\left[\frac{Q_{\rm e}^2 - {\rm i}Q_{\rm e}\!\cdot Q_{\rm m}\,(S-\bar S) + Q_{\rm m}^2\,|S|^2}{S+\bar S}\right] + 128\,c_1\,\log\left[(S+\bar S)^6|\eta(S)|^{24}\right]\ , \tag{2}$$

with the dilaton subject to (1). The first term in this equation corresponds to one-fourth of the horizon area, which, via (1), is affected by the various corrections. The second term represents an extra modification, which explicitly contains the non-holomorphic correction. The above results are invariant under target-space duality and S-duality. As explained above, this was achieved at the price of including non-holomorphic terms, here residing in the $\log(S+\bar S)$ terms, at an intermediate stage of the calculation. Under S-duality the dilaton field transforms in

[1] We have deviated from the notation in [12] and employ the definitions: $Q_{\rm e}^2 = -\langle M, M\rangle$, $Q_{\rm m}^2 = -\langle N, N\rangle$ and $Q_{\rm e}\!\cdot Q_{\rm m} = M\cdot N$. Note that, in the limit of large charges, $Q_{\rm e}^2$ and $Q_{\rm m}^2$ are negative.

the usual manner under $\mathrm{SL}(2,\mathbb{Z})$ and the $\mathrm{SO}(6,22)$ invariant contractions of the charges transform according to the corresponding arithmetic subgroup of $\mathrm{SO}(2,1)$,

$$\begin{aligned} S &\to \frac{a\,S-\mathrm{i}b}{\mathrm{i}c\,S+d}\,, \\ Q_\mathrm{e}^2 &\to a^2\,Q_\mathrm{e}^2+b^2\,Q_\mathrm{m}^2+2\,ab\,Q_\mathrm{e}{\cdot}Q_\mathrm{m}\,, \\ Q_\mathrm{m}^2 &\to c^2Q_\mathrm{e}^2+d^2\,Q_\mathrm{m}^2+2\,cd\,Q_\mathrm{e}{\cdot}Q_\mathrm{m}\,, \\ Q_\mathrm{e}{\cdot}Q_\mathrm{m} &\to ac\,Q_\mathrm{e}^2+bd\,Q_\mathrm{m}^2+(ad+bc)\,Q_\mathrm{e}{\cdot}Q_\mathrm{m}\,. \end{aligned} \tag{3}$$

Here a,b,c,d are integer-valued with $ad-bc=1$, such that they preserve the charge lattice. Observe that the above transformation rules then fix the normalization of the dilaton field.

In string perturbation theory the real part of S becomes large and positive, and one can neglect the exponential terms of the Dedekind eta-function. In that approximation the imaginary part of S equals $\mathrm{Im}\,S=-Q_\mathrm{e}{\cdot}Q_\mathrm{m}/Q_\mathrm{m}^2$ and the real part is determined by a quadratic equation,

$$\tfrac{1}{4}Q_\mathrm{m}^2\left(Q_\mathrm{m}^2+512\,c_1\right)(S+\bar S)^2-\frac{768\,c_1}{\pi}\,Q_\mathrm{m}^2\,(S+\bar S)=Q_\mathrm{e}^2\,Q_\mathrm{m}^2-(Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2\,. \tag{4}$$

Obviously, these perturbative results are affected by the presence of the non-holomorphic corrections. Using (4), we find the following expression for the corresponding entropy,

$$\mathcal{S}_\mathrm{macro}=-2\pi\,\frac{Q_\mathrm{e}^2\,Q_\mathrm{m}^2-(Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2}{Q_\mathrm{m}^2\,(S+\bar S)}+768\,c_1\left[\log(S+\bar S)-1\right]\,. \tag{5}$$

In this paper we will be considering the limit of large charges, where $Q_\mathrm{e}^2\,Q_\mathrm{m}^2-(Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2\gg 1$ and $Q_\mathrm{e}^2+Q_\mathrm{m}^2$ is large and negative. We will consider a uniform scaling of all the charges. The dilaton field will remain finite in that limit; to ensure that it is nevertheless large, one must assume that $|Q_\mathrm{m}^2|$ is sufficiently small as compared to $\sqrt{Q_\mathrm{e}^2\,Q_\mathrm{m}^2-(Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2}$.

In the $N=4$ setting the purely electric or magnetic configurations constitute 1/2-BPS states, whereas the dyonic ones are 1/4-BPS states. In the $N=2$ truncation this distinction disappears. While the results of [12] apply to both cases, the purely electric case was not given much attention at the time. To describe the 1/2-BPS states, we assume Q_e^2 to be large and negative and $Q_\mathrm{m}^2=Q_\mathrm{e}{\cdot}Q_\mathrm{m}=0$ so that we are in a different domain of the charge lattice, where $Q_\mathrm{e}^2Q_\mathrm{m}^2-(Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2=0$. In this case the leading contributions to the entropy and the area will vanish. Consequently, the subleading contributions will now dominate and one finds,

$$\begin{aligned} S+\bar S &\approx \sqrt{Q_\mathrm{e}^2/(128\,c_1)}\,, \\ \mathcal{S}_\mathrm{macro} &\approx 2\,\pi\,\sqrt{128\,c_1\,Q_\mathrm{e}^2}+384\,c_1\log|Q_\mathrm{e}^2|\,. \end{aligned} \tag{6}$$

The leading term in the entropy is now one-half of the area. This is due to the fact that the terms proportional to the square of the curvature contribute to the area and entropy with a relative factor 2, as was already noted in [12], while the 'classical' contribution vanishes. The physical implications of this phenomenon have recently been discussed in a number of papers [14–17].

Let us compare (6) to the asymptotic degeneracy of 1/2-BPS states of heterotic string theory, which is given by

$$d(Q_{\rm e}) = \oint {\rm d}\sigma \, \frac{{\rm e}^{{\rm i}\pi\sigma Q_{\rm e}^2}}{\eta^{24}(\sigma)} \approx \exp\left(4\pi\sqrt{\frac{|Q_{\rm e}^2|}{2}} - \frac{27}{4}\log|Q_{\rm e}^2|\right) , \tag{7}$$

where the integration contour encircles the point $q \equiv \exp(2\pi{\rm i}\sigma) = 0$. We therefore find agreement at leading order in large $|Q_{\rm e}^2|$, provided that $c_1 = -\frac{1}{64}$ as claimed, while the logarithmic corrections fail to agree. The value for c_1 can also be deduced from string-string duality. For type-II string theory compactified on $K3 \times T^2$, c_1 is equal to $-\frac{1}{24\cdot 64}\,\chi$, where χ is the Euler number of $K3$. As the latter is equal to 24, one obtains the same value for c_1. It is worth pointing out that the coefficient of the logarithmic terms (6) is thus equal to -6. The difference with the corresponding term in (7) is precisely the contribution that one obtains from the Gaussian integral when deriving the right-hand side of (7) by a saddle-point approximation. It seems unlikely that this is a coincidence. A recent, extended discussion of this discrepancy can be found in [16].

As stated above, this paper will deal with the comparison between the degeneracy of dyons proposed in [11] and the subleading corrections that are known from the supergravity description [3,12]. One of the main results is that, in the limit of large charges, the degeneracy formula leads to precisely the equations (1) and (2). The paper is organized as follows. In section 2 we introduce the formula for the microscopic dyon degeneracies in $N = 4$ string theory. In section 3 we evaluate its asymptotic behaviour in the limit of large charges and show that it is in precise agreement with the results from the macroscopic description. Finally, in section 4, we present our conclusions. We also discuss the contribution of non-holomorphic terms to the thermodynamic free energy and some aspects related to the 1/2-BPS states. Finally we point out that the attractor equations (1) ensure the stationarity of the expression for the macroscopic entropy (2), written as a function of the dilaton field.

2 Counting dyon states

Quite some time ago, Dijkgraaf, Verlinde and Verlinde proposed a formula for the microscopic degeneracies of dyonic states of $N = 4$ string theory [11]. The degeneracy is expressed in terms of an integral over an appropriate 3-cycle that involves an automorphic form $\Phi_{10}(\Omega)$,

$$d(Q_e, Q_m) = \oint {\rm d}\Omega \, \frac{{\rm e}^{{\rm i}\pi(Q^T\Omega\, Q)}}{\Phi_{10}(\Omega)} , \tag{8}$$

where Ω is the period matrix for a genus-2 Riemann surface; it parametrizes the Sp(2)/U(2) cosets and can be written as a complex, symmetric, two-by-two matrix. In the exponential factor the direct product of the period matrix with the invariant metric of the charge lattice (the latter is suppressed in (8)) is contracted with the charge vector $(Q_{\rm m}, Q_{\rm e})$ comprising the 28 magnetic and 28 electric charges. This formula was conjectured based on the fact that it generalizes the expression (7) for the degeneracies of electric heterotic string states to an

expression that is manifestly covariant with respect to S-duality. In what follows we shall be using the parametrization,

$$\Omega = \begin{pmatrix} \rho & \upsilon \\ \upsilon & \sigma \end{pmatrix} , \qquad Q = \begin{pmatrix} Q_{\rm m} \\ Q_{\rm e} \end{pmatrix} , \tag{9}$$

so that $Q^T \Omega\, Q = \rho\, Q_{\rm m}^2 + \sigma\, Q_{\rm e}^2 + 2\,\upsilon\, Q_{\rm e}{\cdot} Q_{\rm m}$.

The period matrix Ω transforms under $\mathrm{Sp}(2,\mathbb{Z})$ transformations, which can be written as a four-by-four matrix decomposed into four real two-by-two blocks, A, B, C, and D according to,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{with} \qquad \begin{array}{l} A^{\rm T} D - C^{\rm T} B = D\, A^{\rm T} - C\, B^{\rm T} = \mathbf{1}_2 \,, \\ A^{\rm T} C = C^{\rm T} A \,, \qquad B^{\rm T} D = D^{\rm T} B \,. \end{array} \tag{10}$$

In terms of these sub-matrices, Ω transforms as follows under $\mathrm{Sp}(2,\mathbb{Z})$,

$$\Omega \to \Omega' = (A\,\Omega + B)\,(C\,\Omega + D)^{-1} \,. \tag{11}$$

An important related result is, $\Omega - \bar\Omega \to (\Omega\, C^{\rm T} + D^{\rm T})^{-1} (\Omega - \bar\Omega)\, (C\,\bar\Omega + D)^{-1}$.

Modular forms $\Phi_p(\Omega)$ of degree p transform under the modular group $\mathrm{Sp}(2,\mathbb{Z})$ according to

$$\Phi_p(\Omega') = \det(C\,\Omega + D)^p\, \Phi_p(\Omega) \,. \tag{12}$$

These are holomorphic functions over the Siegel half-space, defined by $\det(\Omega - \bar\Omega) < 0$. The modular form appearing in (8) is the unique cusp form of degree 10. It is proportional to the square of the Siegel cusp form $\Delta_5(\Omega)$, which is of degree 5 and has a non-trivial multiplier system (i.e., there are extra sign factors in (12) depending on the particular $\mathrm{Sp}(2,\mathbb{Z})$ element). The cusp form can be defined as a product over all even theta-constants. From its behaviour under modular transformations, it follows that it can be defined as a Fourier series with unique coefficients (see, e.g. [18]),

$$\Delta_5(\Omega) \;=\; \sum_{\{k,l,m\}} f(k,l,m)\, \exp\left[\mathrm{i}\pi(k\,\rho + l\,\sigma + m\,\upsilon)\right] \,, \tag{13}$$

where the sum extends over $k,l,m = 1 \bmod 2$ with $4kl - m^2 > 0$ and $k,l > 0$. The $f(k,l,m)$ are integral coefficients; for instance, one has $f(1,1,1) = -f(1,1,-1) = 64$. Obviously, a corresponding expansion exists for Φ_{10}.[2]

[2] An alternative form of the Fourier series involves products,

$$\Phi_{10}(\Omega) = \left[\tfrac{1}{64}\Delta_5(\Omega)\right]^2 = q_\rho q_\sigma q_\upsilon \prod_{\{k,l,m\}} (1 - q_\rho^{\,k}\, q_\sigma^{\,l}\, q_\upsilon^{\,m})^{c(kl,m)} \,,$$

where $q_\rho = \exp(2\pi\mathrm{i}\rho)$, $q_\sigma = \exp(2\pi\mathrm{i}\sigma)$ and $q_\upsilon = \exp(2\pi\mathrm{i}\upsilon)$. Here the product extends over integers k,l,m, with $k,l \geq 0$, or, when $k = l = 0$, with $m < 0$. The constants $c(kl,m)$ depend only on $4kl - m^2$ and are related to the elliptic genus of $K3$; they vanish for $4kl - m^2 < -1$.

The cusp form has single zeroes; one is at $\upsilon = 0$ and the other ones are in the $\mathrm{Sp}(2,\mathbb{Z})$ image of $\upsilon = 0$. The zero at $\upsilon = 0$ is obvious from the relation,

$$\Phi_{10}(\Omega) \approx \upsilon^2\, \eta^{24}(\rho)\, \eta^{24}(\sigma)\,, \tag{14}$$

which, for instance, follows from the representation of $\Phi_{10}(\Omega)$ in terms of even theta-constants, and which we shall be using later. Here we note in passing that $\Phi_{10}(\Omega)$ is an even function of υ, as follows from (12) by applying a transformation with $A = D = \mathrm{diag}\,(1,-1)$ and $B = C = 0$. The zeroes emerge as poles in the integrand of (8) and therefore an integral over a 3-cycle that encloses such a pole will correspond to a particular coefficient in the Fourier series for the inverse of Φ_{10}. However, the poles are located in the interior of the Siegel half-space and not just at its boundary. Therefore the choice of the 3-cycles in (8) is subtle, just as the corresponding definition of the coefficients of the Fourier series of $(\Phi_{10})^{-1}$, as one could be picking up extra finite residues when moving the cycle through the Siegel half-space. This aspect should be borne in mind when considering the S-duality covariance of the expression (8).

Formally, the S-duality covariance of (8) follows from the fact that the effect of the transformation (3) of the charges can be compensated for by a special $\mathrm{Sp}(2,\mathbb{Z})$ transformation on the period matrix, $\Omega \to A\,\Omega\,A^{\mathrm{T}}$ (possibly up to integer real shifts associated with the sub-matrix B). This corresponds to taking $D^{-1} = A^{\mathrm{T}}$, and $C = 0$; choosing $A = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$, with $ad - bc = 1$, induces the required $\mathrm{SO}(2,1)$ transformations of (ρ,σ,υ),

$$\begin{aligned}
\rho &\;\to\; a^2\,\rho + b^2\,\sigma - 2\,ab\,\upsilon\,, \\
\sigma &\;\to\; c^2\rho + d^2\,\sigma - 2\,cd\,\upsilon\,, \\
\upsilon &\;\to\; -\,ac\,\rho - bd\,\sigma + (ad+bc)\upsilon\,.
\end{aligned} \tag{15}$$

The fact that the automorphic form Φ_{10} is invariant under this subgroup of $\mathrm{Sp}(2,\mathbb{Z})$ then formally ensures the S-duality covariance of (8). A more rigorous argument along these lines should in principle yield the S-duality invariant charge lattice, but we are not aware of such a result in the literature.

In the next section we will be studying the large charge limit of (8) by first picking out the residue from the integral over υ followed by a saddle-point approximation to perform the remaining integrals in ρ and σ. Here the stationarity requirement of the saddle-point method will implictly deal with the issue of choosing the appropriate 3-cycles.

3 Asymptotic density of dyon states

In [11] it is argued that, in the limit of large charges, the leading behaviour of the degeneracy of dyon states is determined by poles associated with the rational quadratic divisor,

$$\mathcal{D} = \upsilon + \rho\sigma - \upsilon^2 = 0\,. \tag{16}$$

Subsequently, it was shown that the leading-order contribution agrees with the macroscopic black hole entropy based on the area law. We expect that the subleading contributions can be

extracted from the same pole terms, up to exponentially suppressed contributions. In order to compute the subleading contributions, we need to know the form of the automorphic form $\Phi_{10}(\Omega)$ in the vicinity of the divisor (16). This can be obtained from the degeneracy limit $\upsilon \to 0$, for which the automorphic form $\Phi_{10}(\Omega)$ has the behaviour already indicated in (14),

$$\frac{1}{\Phi_{10}(\Omega)} \longrightarrow \frac{1}{\upsilon^2}\,\frac{1}{\eta^{24}(\rho)\,\eta^{24}(\sigma)} + \mathcal{O}(\upsilon^0)\,. \tag{17}$$

The divisor $\upsilon = 0$ is related to the divisor (16) by a $\mathrm{Sp}(2,\mathbb{Z})$ transformation given by $-B = C = \mathbf{1}_2$, $D = 0$, and $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which yields

$$\begin{pmatrix} \rho & \upsilon \\ \upsilon & \sigma \end{pmatrix} \longrightarrow \begin{pmatrix} \rho' & \upsilon' \\ \upsilon' & \sigma' \end{pmatrix} = \frac{1}{\det\Omega}\begin{pmatrix} -\sigma & \upsilon + \rho\sigma - \upsilon^2 \\ \upsilon + \rho\sigma - \upsilon^2 & -\rho \end{pmatrix}\,. \tag{18}$$

This transformation determines an expansion of $\Phi_{10}(\Omega)$ similar to (17) as $\mathcal{D} \to 0$,

$$\frac{1}{\Phi_{10}(\Omega)} = \frac{\det(\Omega)^{10}}{\Phi_{10}(\Omega')} \longrightarrow \frac{1}{\mathcal{D}^2}\,\frac{\det(\Omega)^{12}}{\eta^{24}(\rho')\,\eta^{24}(\sigma')} + \mathcal{O}(\mathcal{D}^0)\,. \tag{19}$$

The arguments of the Dedekind eta-functions are given by (18),

$$\rho' = -\frac{\sigma}{\rho\sigma - \upsilon^2}\,, \qquad \sigma' = -\frac{\rho}{\rho\sigma - \upsilon^2}\,. \tag{20}$$

The contributions from other divisors will be exponentially suppressed and we can now insert expression (19) into (8) and evaluate the contour integral for υ around the poles $\upsilon_\pm = \frac{1}{2} \pm \Lambda$, where we have defined $\Lambda = \sqrt{\frac{1}{4} + \rho\sigma}$. Introducing $\gamma' = -1/\rho'$, we find that the integrand for the remaining integral over ρ and σ becomes

$$\Delta(\rho,\sigma)\,\exp\left[\mathrm{i}\pi Q_\mathrm{e}{\cdot}Q_\mathrm{m} + \mathrm{i}\pi X(\rho,\sigma)\right]\,, \tag{21}$$

where

$$\begin{aligned} X(\rho,\sigma) &= \rho\,Q_\mathrm{m}^2 + \sigma\,Q_\mathrm{e}^2 \pm 2\,\Lambda\,Q_\mathrm{e}{\cdot}Q_\mathrm{m} + \frac{12\alpha}{\mathrm{i}\pi}\log\sigma - \frac{24\alpha}{\mathrm{i}\pi}\log\eta(\sigma') - \frac{24\alpha}{\mathrm{i}\pi}\log\eta(\gamma')\,, \\ \Delta(\rho,\sigma) &= \frac{1}{4\Lambda^2}\left[2\pi\mathrm{i}\,Q_\mathrm{e}{\cdot}Q_\mathrm{m} \mp \frac{1}{\Lambda} + 48\alpha\left(\sigma'\frac{\mathrm{d}\log\eta(\sigma')}{\mathrm{d}\sigma'} - \gamma'\frac{\mathrm{d}\log\eta(\gamma')}{\mathrm{d}\gamma'}\right)\right]\,. \end{aligned} \tag{22}$$

The term $\mathrm{i}\pi Q_\mathrm{e}{\cdot}Q_\mathrm{m}$ in (21) represents an overall sign factor as $Q_\mathrm{e}{\cdot}Q_\mathrm{m}$ is expected to take integer values, so that we will drop it in the following. Furthermore, we have introduced the parameter α, which is given by $\alpha = 1$, in order to keep track of the terms coming from the Dedekind eta-functions. This parameter is the counterpart to $-64c_1$ in (2). In order to arrive at this result, we have used the modular properties of the Dedekind eta-function to express $\eta^{24}(\rho')$ in terms of $\eta^{24}(-1/\rho')$, which gives rise to the term $12\alpha\log\sigma$ in $\mathrm{i}\pi X(\rho,\sigma)$. In doing so the factors $\det(\Omega)$ cancel.

It is instructive to express the integrand solely in terms σ' and γ'. To this extent we note the following identities valid on the divisor (16),

$$\upsilon_\pm = \tfrac{1}{2} \pm \Lambda = \frac{\gamma'}{\sigma' + \gamma'}\,, \quad \rho = \frac{\sigma'\gamma'}{\sigma' + \gamma'}\,, \quad \sigma = -\frac{1}{\sigma' + \gamma'}\,. \tag{23}$$

Substituting these expressions in (22), the exponent takes the suggestive form

$$\mathrm{i}\pi X(\sigma,\rho) \quad = \quad -\pi \left[\frac{Q_{\mathrm{e}}^2 + (\sigma'-\gamma')Q_{\mathrm{e}}\cdot Q_{\mathrm{m}} - \sigma'\gamma' Q_{\mathrm{m}}^2}{-\mathrm{i}(\sigma'+\gamma')}\right] - 2\alpha \log\left[(\sigma'+\gamma')^6 \eta(\sigma')^{12}\eta(\gamma')^{12}\right] , \tag{24}$$

which holds for both poles at $\upsilon = \upsilon_\pm$. At this point we observe a remarkable fact: if one identifies,

$$\sigma' = \mathrm{i}\bar S\,, \qquad \gamma' = \mathrm{i}S\,, \tag{25}$$

in the expression (24), it precisely coincides with the macroscopic entropy formula (2) presented in section 1. Also the arguments of the Dedekind eta-functions match: the functions $\eta(S)$ and $\eta(\bar S)$ that appear in (1) and (2) are functions of the argument $q = \mathrm{e}^{-2\pi S}$ and $q = \mathrm{e}^{-2\pi \bar S}$, respectively, while in the microscopic approach $\eta(\sigma')$ and $\eta(\gamma')$ are functions of $q = \mathrm{e}^{2\pi\mathrm{i}\sigma'}$ and $q = \mathrm{e}^{2\pi\mathrm{i}\gamma'}$, respectively.

Before proceeding let us first consider some consequences of this surprising match. First of all, the identification implies that σ' is equal to minus the complex conjugate of γ', and therefore the expression (24) is real. Secondly, one may wonder what the consequences are for S-duality. To investigate this, let us determine the S-duality transformations on ρ, σ and $\upsilon_\pm$ as induced by the S-duality variations of S, through (23). A simple calculation yields the following result,

$$\begin{aligned}
\rho &\;\rightarrow\; a^2\,\rho + b^2\,\sigma - 2\,ab\,\upsilon_\pm + ab\,,\\
\sigma &\;\rightarrow\; c^2\rho + d^2\,\sigma - 2\,cd\,\upsilon_\pm + cd\,,\\
\upsilon_\pm &\;\rightarrow\; -\,ac\,\rho - bd\,\sigma + (ad+bc)\upsilon_\pm - bc\,.
\end{aligned} \tag{26}$$

Hence, these transformations coincide with the transformations (15), up to translations by integers. In fact, they constitute the subgroup of $\mathrm{Sp}(2,\mathbb{Z})$ that leaves the divisor (16) invariant. This explains why they apply irrespective of which pole one chooses. Based on these observations, the identification (25) seems to be a very sensible one indeed.

The remaining two integrals associated with the 3-cycle will be carried out in a saddle-point approximation. Note, however, that the integrand (21) also contains a contribution from the factor $\Delta(\rho,\sigma)$. Of course, both $X(\rho,\sigma)$ and $\Delta(\rho,\sigma)$ will enter in the saddle-point evaluation of the integral and in principle, one should determine the saddle-point values of ρ and σ from the extremality conditions of the complete integrand. Nevertheless, we will treat the two pieces $X(\rho,\sigma)$ and $\Delta(\rho,\sigma)$ of the integrand (21) separately. While $\Delta(\rho,\sigma)$ does contribute (logarithmically) to the saddle-point value of the integrand, we initially neglect $\Delta(\rho,\sigma)$ when determining the saddle-point. Hence we will be expanding the integrand around an approximate extremal point and after performing the integrations there will be additional contributions involving derivatives of $\log(\Delta(\rho,\sigma))$. As we shall see, these contributions are suppressed by inverse powers of derivatives of $X(\rho,\sigma)$ and the approximation is reliable because the derivatives of $X(\rho,\sigma)$ contain terms proportional to the charges, unlike the derivatives of $\log(\Delta(\rho,\sigma))$.

The saddle-point equations derived from $\exp(\mathrm{i}\pi X(\rho,\sigma))$ are given by

$$\mathrm{i}\pi\partial_\rho X = \mathrm{i}\pi Q_\mathrm{m}^2 + 2\,\mathrm{i}\,\pi Q_\mathrm{e}\cdot Q_\mathrm{m}\,\frac{1}{\sigma'-\gamma'} + 24\alpha\,\frac{\sigma'+\gamma'}{\sigma'-\gamma'}\left[\frac{\mathrm{d}\log\eta(\sigma')}{\mathrm{d}\sigma'} - \frac{\mathrm{d}\log\eta(\gamma')}{\mathrm{d}\gamma'}\right] = 0\;, \qquad (27)$$

and

$$\begin{aligned}\mathrm{i}\pi\partial_\sigma X = {}& \mathrm{i}\pi Q_\mathrm{e}^2 - 2\,\mathrm{i}\,\pi(Q_\mathrm{e}\cdot Q_\mathrm{m})\,\frac{\sigma'\gamma'}{\sigma'-\gamma'} \\ & - 12\alpha(\sigma'+\gamma') - 24\alpha\,\frac{\sigma'+\gamma'}{\sigma'-\gamma'}\left[\sigma'^2\,\frac{\mathrm{d}\log\eta(\sigma')}{\mathrm{d}\sigma'} - \gamma'^2\frac{\mathrm{d}\log\eta(\gamma')}{\mathrm{d}\gamma'}\right] = 0\;. \qquad (28)\end{aligned}$$

Note that $\mathrm{i}\pi X(\rho,\sigma)$ and the saddle-point equations (27) and (28) derived from it hold irrespective of the pole at $\upsilon_\pm$ that has been selected in the initial contour integral. The only dependence resides in the relation between σ',γ' and ρ,σ, specified by (23). But since the saddle-point equations are identical for both poles, the choice of the pole is irrelevant.

With the identification (25) the saddle-point equations (27) and (28) precisely coincide with the non-holomorphic attractor equations (1). We conclude that a solution to the saddle-point equations is provided by

$$\sigma'|_0 = \mathrm{i}\bar{S}\;, \quad \gamma'|_0 = \mathrm{i}S\;, \qquad (29)$$

where S is subject to (1).For this solution, $\mathrm{i}\pi X$ given in (24) is precisely equal to the macroscopic black hole entropy! Although it is conceivable that the condition $\gamma' = -\bar{\sigma}'$, which is implied by (25), can be relaxed and other saddle-point values than those dictated by the attractor equations (1) can be found, we believe this to be unlikely. Firstly, for the leading, α-independent terms in (24) to be real, $\sigma'+\gamma'$ must be imaginary and $\sigma'-\gamma'$ real. This implies that $\gamma' = -\bar{\sigma}'$. Secondly, we prove (29) by direct calculation in the limit where σ' and γ' have large and positive imaginary parts. In this limit, we can expand the Dedekind eta-functions and solve the saddle-point equations for $\sigma'-\gamma'$ and $\sigma'+\gamma'$. The equations (27) and (28) simplify and one finds

$$\sigma'-\gamma' = -\frac{2\,Q_\mathrm{e}\cdot Q_\mathrm{m}}{Q_\mathrm{m}^2}\;. \qquad (30)$$

Inserting this expression into (28) we find the following equation for $\sigma'+\gamma'$,

$$-\tfrac{1}{4}Q_\mathrm{m}^2(Q_\mathrm{m}^2-8\alpha)(\sigma'+\gamma')^2 + \frac{12\alpha}{\mathrm{i}\pi}Q_\mathrm{m}^2(\sigma'+\gamma') = Q_\mathrm{e}^2\,Q_\mathrm{m}^2 - (Q_\mathrm{e}\cdot Q_\mathrm{m})^2\;. \qquad (31)$$

This is a quadratic equation which uniquely determines the value of $\sigma'+\gamma'$ in terms of the charges. A comparison of (31) with (4) shows that the saddle-point values for $-\mathrm{i}(\sigma'+\gamma')$ and $\mathrm{i}(\sigma'-\gamma')$ are precisely the attractor values for the real and imaginary parts of the dilaton S. Hence, we are necessarily led to $\sigma'-\gamma' = -\mathrm{i}(S-\bar{S})$ and $\sigma'+\gamma' = \mathrm{i}(S+\bar{S})$ in accord with the identification (25).

Let us return to saddle-point approximation for the remaining integrals. As stressed above, we now have to include the factor $\Delta(\rho,\sigma)$, which has the following form,

$$\log\Delta(\rho,\sigma) = \log\left[\frac{2\mathrm{i}\pi\, Q_{\rm e}\!\cdot\! Q_{\rm m}\,(\sigma'+\gamma')^2}{(\sigma'-\gamma')^2}\right]$$
$$+\log\left[1+\frac{1}{\mathrm{i}\pi\, Q_{\rm e}\!\cdot\! Q_{\rm m}}\left(\sigma'\frac{\mathrm{d}}{\mathrm{d}\sigma'}-\gamma'\frac{\mathrm{d}}{\mathrm{d}\gamma'}\right)\left(\log\left[\sigma'-\gamma'\right]+2\,\alpha\,\log\left[\eta^{12}(\sigma')\,\eta^{12}(\gamma')\right]\right)\right]\,. \tag{32}$$

Furthermore we need the expression for the matrix of the second derivatives of $\mathrm{i}\pi X(\rho,\sigma)$, which takes the form

$$\frac{\partial^2\left(\mathrm{i}\pi\, X(\rho,\sigma)\right)}{\partial^2(\rho,\sigma)} = -\,\frac{2\mathrm{i}\pi Q_{\rm e}\!\cdot\! Q_{\rm m}(\sigma'+\gamma')}{(\sigma'-\gamma')^3}\begin{pmatrix}-2 & \sigma'^2+\gamma'^2\\ \sigma'^2+\gamma'^2 & -2\,\sigma'^2\,\gamma'^2\end{pmatrix}+\cdots\,, \tag{33}$$

where we suppressed terms that do not depend explicitly on the charges.

With these expressions we can complete the saddle-point approximation. First, we note that the derivatives of $\log(\Delta(\rho,\sigma))$ are of order $(Q_{\rm e}\!\cdot\! Q_{\rm m})^0$, whereas the leading term of the matrix of second-order derivatives of $\mathrm{i}\pi X(\rho,\sigma)$ is proportional to $Q_{\rm e}\!\cdot\! Q_{\rm m}$. Furthermore, we have to perform a two-dimensional integral over the real values of ρ and σ. Consequently, the result of the saddle-point approximation yields

$$\mathcal{S}_{\rm micro} = \mathrm{i}\pi X\big|_0 + \log\Delta\big|_0 - \tfrac{1}{2}\log\det\frac{\partial^2(\mathrm{i}\pi X(\rho,\sigma))}{\partial^2(\rho,\sigma)}\bigg|_0\,, \tag{34}$$

up to terms that behave inversely proportional to $Q_{\rm e}Q_{\rm m}$. Note, however, that the second and third term depend only on $Q_{\rm e}\!\cdot\! Q_{\rm m}$, whereas the first term depends on all three combinations $Q_{\rm e}^2$, $Q_{\rm m}^2$ and $Q_{\rm e}\!\cdot\! Q_{\rm m}$ and it is known to be invariant under S-duality. Therefore, one expects that the contributions from the second and third term will cancel. This is indeed the case: computing the last term of (34), we find that it is given by

$$-\tfrac{1}{2}\log\det\frac{\partial^2(\mathrm{i}\pi X(\rho,\sigma))}{\partial^2(\rho,\sigma)}\bigg|_0 = -\log\left[\frac{2\pi\mathrm{i}Q_{\rm e}\!\cdot\! Q_{\rm m}\,(\sigma'+\gamma')^2}{(\sigma'-\gamma')^2}\right]+\mathcal{O}(1/Q_{\rm e}\!\cdot\! Q_{\rm m})\,. \tag{35}$$

This term cancels exactly against the first term in (32). Likewise, the attractor equations, which correspond to the saddle-point equations (27) and (28), are only modified by terms that are inversely proportional to $Q_{\rm e}\!\cdot\! Q_{\rm m}$.

We have thus verified that $d(Q_{\rm e},Q_{\rm m})$ defined in (8) is equal to $\exp(\mathcal{S}_{\rm macro})$ for large charges with nonvanishing $Q_{\rm e}\!\cdot\! Q_{\rm m}$. Here $\mathcal{S}_{\rm macro}$ represents the macroscopic entropy given in (2), subject to the attractor equations (1). We expect that the same result can be established for the case $Q_{\rm e}\!\cdot\! Q_{\rm m}=0$. Consequently, we have shown that the dyon degeneracy formula (8) leads precisely to the results of [12] summarized in equations (1) and (2).

4 Conclusions and Outlook

In the previous section we have established that the formula (2) for the macroscopic entropy is in agreement with the microscopic degeneracy of dyons proposed in [11]. The former

includes non-holomorphic terms which are crucial for obtaining an S-duality invariant result. Here we have demonstrated that these non-holomorphic terms are precisely captured by the microscopic counting.

Let us first discuss the structure of the non-holomorphic corrections, also in the light of the observation in [10] that the formula for the black hole entropy formula can be reinterpreted as the Legendre transform of the black hole free energy. The entropy formula of [3] is based on a supersymmetric Wilsonian effective action, which for $N = 2$ supergravity is encoded in a homogeneous, holomorphic function of projectively defined quantities. In the formalism of [3] a convenient set of complex variables was found, denoted by Y^I and Υ. The variable Υ is associated with an extra chiral supermultiplet related to the Weyl multiplet of conformal supergravity and its presence in the holomorphic function gives rise to terms in the effective action proportional to the square of the Riemann curvature. The relevant function for the heterotic case takes the following form [12],

$$F(Y,\Upsilon) = -\frac{Y^1\, Y^a \eta_{ab} Y^b}{Y^0} + F^{(1)}(Y^1/Y^0)\,\Upsilon\,, \tag{36}$$

where $F^{(1)}$ is some function of the dilaton field $S = -\mathrm{i}Y^1/Y^0$. The entropy formula for generic homogeneous and holomorphic functions F can be written as

$$\mathcal{S}_{\mathrm{macro}} = \pi\left[p^I F_I(Y,\Upsilon) - q_I Y^I - 256\,\mathrm{Im}(F_\Upsilon(Y,\Upsilon))\right]_{\Upsilon=-64}\,, \tag{37}$$

where the value of -64 represents the value of Υ taken at the horizon. Likewise, the electric and magnetic charges q_I and p^I, respectively, determine the horizon values of the Y^I according to the attractor equations (using the horizon value $\Upsilon = -64$),

$$Y^I - \bar{Y}^I = \mathrm{i}p^I\,, \qquad F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon}) = \mathrm{i}q_I\,, \tag{38}$$

where $F_I = \partial F/\partial Y^I$ and $F_\Upsilon = \partial F/\partial\Upsilon$. The first two terms in (37), which are real by virtue of (38), represent one-fourth of the black hole area, while the last term proportional to F_Υ represents the deviation of the area law.

In [12] the function $F^{(1)}(S)$ in (36) was determined by requiring target-space duality and S-duality invariance. For achieving consistency with string perturbation theory one must introduce non-holomorphic terms. The approach that was followed for deriving these terms was somewhat ad hoc. It was first assumed that the attractor equations (38) for Y^I and F_I still hold, but that these quantities contain non-holomorphic terms, corresponding to

$$F^{(1)}(S,\bar{S}) = -\frac{\mathrm{i}c_1}{\pi}\log\left[(S+\bar{S})^6\eta^{12}(S)\right]\ . \tag{39}$$

However, when substituting this modification into the entropy formula (37) it did not produce an S-duality invariant result and one had to introduce yet another term equal to $-128\,c_1\log(S+\bar{S})^6$. The combined result of these modifications is concisely summarized in (1) and (2). As we have already explained in the introduction, the presence of non-holomorphic corrections is to be expected in view of the fact that the Wilsonian action, which is based on holomorphicity, does in general not reflect the symmetries of the theory.

In [10] the result of the holomorphic case was reformulated in terms of a real function and it is of interest to see how the non-holomorphic terms will manifest themselves in that formulation. This is a priori not completely obvious as both the holomorphicity and the homogeneity of the function F were used in the derivation. In [10] the Y^I were expressed in terms of the magnetic charges p^I and (real) electrostatic potentials ϕ^I at the horizon,

$$Y^I = \frac{\phi^I}{2\pi} + \frac{\mathrm{i}p^I}{2}\,, \tag{40}$$

so that the first set of attractor equations (38) is already incorporated. The remaining attractor equations (38) and the entropy (37) can then be written as follows,

$$\begin{aligned} q_I &= \frac{\partial\mathcal{F}(\phi,p)}{\partial\phi^I}\,, \\ \mathcal{S}_{\rm macro}(p,q) &= \mathcal{F}(\phi,p) - \phi^I\,\frac{\partial\mathcal{F}(\phi,p)}{\partial\phi^I}\,, \end{aligned} \tag{41}$$

where the real function $\mathcal{F}(\phi,p)$ is defined by

$$\mathcal{F}(\phi,p) = 4\pi\,\mathrm{Im}[\,F(Y,\Upsilon)]_{\Upsilon=-64}\,. \tag{42}$$

This function is related to the entropy by a Legendre transformation and therefore it will be called the thermodynamic free energy. The homogeneity of the function $F(Y,\Upsilon)$ was crucial for deriving (41) and (42) and use was made of the corresponding identity

$$Y^I F_I(Y,\Upsilon) + 2\,\Upsilon F_\Upsilon(Y,\Upsilon) = 2\,F(Y,\Upsilon)\,. \tag{43}$$

To study the effect of (possibly non-holomorphic) corrections, let us simply modify $\mathcal{F}$ according to

$$\widehat{\mathcal{F}}(\phi,p) = 4\pi\,\mathrm{Im}[\,F(Y,\Upsilon)] + 4\pi\,\Omega(Y,\bar Y,\Upsilon,\bar\Upsilon)\,, \tag{44}$$

where for the moment we refrain from setting $\Upsilon=-64$. Here Ω denotes a real function. We then assume that (41) remains valid with $\mathcal{F}(\phi,p)$ replaced by $\widehat{\mathcal{F}}(\phi,p)$, where (40) still defines the magnetic charges and electrostatic potentials. The electric charges are thus defined by

$$\frac{\partial}{\partial Y^I}(F+2\mathrm{i}\,\Omega) - \frac{\partial}{\partial\bar Y^I}(\bar F - 2\mathrm{i}\,\Omega) = \mathrm{i}\,\frac{\partial\widehat{\mathcal{F}}}{\partial\phi^I} = \mathrm{i}q_I\,, \tag{45}$$

while the entropy can now be written as

$$\begin{aligned} \mathcal{S}_{\rm macro} &= \pi\Big[p^I\,\frac{\partial}{\partial Y^I}(F+2\mathrm{i}\,\Omega) - q_I\,Y^I - 2\mathrm{i}\,(\Upsilon\,F_\Upsilon - \bar\Upsilon\,\bar F_{\bar\Upsilon}) \\ &\qquad +2\Big(2 - Y^I\frac{\partial}{\partial Y^I} - \bar Y^I\frac{\partial}{\partial\bar Y^I}\Big)\,\Omega\Big]\,. \end{aligned} \tag{46}$$

In deriving this result we used (45) and the homogeneity and holomorphicity of $F(Y,\Upsilon)$.

Furthermore, if we assume that Ω is a homogeneous function of degree 2, so that

$$2\,\Omega - Y^I\frac{\partial\Omega}{\partial Y^I} - \bar Y^I\frac{\partial\Omega}{\partial\bar Y^I} = 2\,\Upsilon\frac{\partial\Omega}{\partial\Upsilon} + 2\,\bar\Upsilon\frac{\partial\Omega}{\partial\bar\Upsilon}\,, \tag{47}$$

we can make direct contact with (37) via the substitutions

$$\begin{aligned} F_I(Y,\Upsilon) &\longrightarrow F_I(Y,\Upsilon) + 2\mathrm{i}\,\frac{\partial\Omega(Y,\bar Y,\Upsilon,\bar\Upsilon)}{\partial Y^I}\,,\\ F_\Upsilon(Y,\Upsilon) &\longrightarrow F_\Upsilon(Y,\Upsilon) + 2\mathrm{i}\,\frac{\partial\Omega(Y,\bar Y,\Upsilon,\bar\Upsilon)}{\partial \Upsilon}\,. \end{aligned} \tag{48}$$

So far we have not made any assumption about the holomorphic behaviour of Ω. Clearly, when Ω can be written as the imaginary part of a holomorphic function $\Omega(Y,\Upsilon)$, then we recover the previous result with the function $F(Y,\Upsilon)$ replaced by $F(Y,\Upsilon)+\Omega(Y,\Upsilon)$. However, the result can equally well be applied to more general corrections. For instance, assuming that $F(Y,\Upsilon)$ is just equal to the classical result, characterized by the Υ-independent first term in (36), we find the complete result (1) and (2) by substituting the following expression for Ω,

$$\Omega(Y,\bar Y,\Upsilon,\bar\Upsilon) = -\frac{c_1}{2\pi}\left[\Upsilon\log\eta^{12}(S) + \bar\Upsilon\log\eta^{12}(\bar S) + \tfrac{1}{2}(\Upsilon+\bar\Upsilon)\log(S+\bar S)^6\right]\,. \tag{49}$$

At this point we are thus able to give the corresponding expression for $\widehat{\mathcal{F}}(\phi,p)$, where now we have substituted $\Upsilon = -64$,

$$\begin{aligned} \widehat{\mathcal{F}}(\phi,p) &= (S+\bar S)\,\eta_{ab}\left(\frac{\pi\,p^ap^b}{2} - \frac{\phi^a\phi^b}{2\pi}\right) - \mathrm{i}(S-\bar S)\,\eta_{ab}\,p^a\phi^b \\ &\quad +128\,c_1\log\left[(S+\bar S)^6\,|\eta(S)|^{24}\right]\,, \end{aligned} \tag{50}$$

where

$$S = -\mathrm{i}\,\frac{\phi^1+\mathrm{i}\pi p^1}{\phi^0+\mathrm{i}\pi p^0}\,. \tag{51}$$

Thus we see that the real function (50) associated to the macroscopic entropy (2) contains non-holomorphic terms, and that these appear in precisely the combination that corresponds to the S-duality invariant physical coupling function of the R^2-terms [13]. It would be interesting to compare this real function with the one obtained by performing the Laplace transform of the microscopic degeneracy formula (8). The non-holomorphic corrections in (50) are presumably related to the difference between a Legendre and a Laplace transform and associated to integrating out the moduli fields ϕ^a of the $\mathrm{SO}(6,22)$ coset.

Finally, we comment briefly on the case of purely electric black holes. These configurations constitute 1/2-BPS states and have recently been reanalyzed in [14,16], both from the microscopic and macroscopic perspective. We would like to point out that it seems difficult to match the microscopic and the macroscopic results at the non-perturbative level. To illustrate this, let us compare the attractor equations (1) for purely electric black holes (at the non-perturbative level, the dilaton must be real in this case),

$$\mathrm{i}\pi Q_{\mathrm{e}}^2 = 24\,\mathrm{i}\,S - 96\,S^2\,\frac{\mathrm{d}\log\eta(\mathrm{i}S)}{\mathrm{d}(\mathrm{i}S)}\,, \tag{52}$$

with the saddle-point equation for the variable σ derived from the degeneracy formula (7),

$$\mathrm{i}\pi Q_{\mathrm{e}}^2 = 12\gamma + 24\,\gamma^2\,\frac{\mathrm{d}\log\eta(\gamma)}{\mathrm{d}\gamma}\,, \tag{53}$$

where we have redefined $\gamma = -1/\sigma$. In order to facilitate the comparison, we use that the $\eta(\tau)$ in this paragraph is always understood as a power series in $q = \exp(2\pi \mathrm{i}\tau)$. At the perturbative level where S is large, both equations agree upon an identification $\gamma = 2\,\mathrm{i}\,S$, which is presumably related to an observation made in [14] about the dilaton normalization. Contrary to the dyonic case, this identification fails at the non-perturbative level, since matching the arguments of the Dedekind eta-functions would require $\gamma = \mathrm{i}S$. Furthermore, there is no way to achieve a non-perturbative matching by rescaling the charges Q_e^2 and/or the dilaton S. The reason for this mismatch is not known to us at present. Note that in this case the limit of large charges is taken in the sublattice characterized by $Q_\mathrm{e}^2\,Q_\mathrm{m}^2 - (Q_\mathrm{e}{\cdot}Q_\mathrm{m})^2 = 0$, which is unrelated to the sublattice of the dyonic configurations.

We close with the following observation. As shown in this paper the attractor equations (1) are identical to the saddle-point equations which result from the stationarity of the quantity X given in (24). This suggests to view the attractor equations (1) as the conditions that ensure that the expression (2) for the macroscopic entropy, written as a function of the dilaton field S, takes an extremal value at the horizon. The significance of this result is not entirely clear to us, as it depends on the way in which we have written the entropy formula. Namely, by using the attractor equations we could write the expression (2) into an alternative form which takes the same value at the horizon, but for which this stationarity condition would not hold.

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